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# PRINCIPAL SERIES WHITTAKER FUNCTIONS ON $GL(3, \mathbb{C})$

MIKI HIRANO AND TAKAYUKI ODA

## 1. INTRODUCTION

The precise analytic properties of Whittaker functions are utilized in the study of the Fourier expansions of automorphic forms and their related topics such as  $L$ -functions. In this note, we give explicit formulas for (general) principal series Whittaker functions on  $GL(3, \mathbb{C})$ . Also, we give a propagation formula which is an expression of the Whittaker functions on  $GL(3, \mathbb{C})$  via those on  $GL(2, \mathbb{C})$ : This is an analogue of the recent result of Ishii-Stade [5] for the class one cases.

This note is based on our recent paper [3]. See it for details.

## 2. DEFINITION OF WHITTAKER FUNCTIONS

Let  $G = NAK$  be an Iwasawa decomposition of a real reductive group  $G$ . For an irreducible admissible representation  $(\pi, H_\pi)$  of  $G$ , we choose a  $K$ -type  $(\tau^*, V_{\tau^*})$  in  $\pi$  which occurs with multiplicity one and fix an injective  $K$ -homomorphism  $i \in \text{Hom}_K(\tau^*, \pi|_K)$ . Here  $(\tau^*, V_{\tau^*})$  means the contragredient representation of  $(\tau, V_\tau)$ . Moreover, take a non-degenerate character  $\eta$  of  $N$ . Let us consider the intertwining space

$$\mathcal{I}_{\eta, \pi} = \text{Hom}_{(\mathfrak{g}_\mathbb{C}, K)}(\pi, C^\infty \text{Ind}_N^G(\eta))$$

between  $(\mathfrak{g}_\mathbb{C}, K)$ -modules consisting of all  $K$ -finite vectors, where  $C^\infty \text{Ind}_N^G(\eta)$  is the induced representation of  $G$  from  $\eta$  as  $C^\infty$ -induction. For each  $T \in \mathcal{I}_{\eta, \pi}$ , we define a  $V_\tau$ -valued function  $T_i$  on  $G$  by

$$T(i(v^*))(g) = \langle v^*, T_i(g) \rangle, \quad v^* \in V_{\tau^*}, \quad g \in G.$$

Here  $\langle \cdot, \cdot \rangle$  is the canonical bilinear form on  $V_{\tau^*} \times V_\tau$ . The function  $T_i$  means a restriction of  $T \in \mathcal{I}_{\eta, \pi}$  to  $K$  and satisfies

$$T_i(n g k) = \eta(n) \tau(k)^{-1} T_i(g), \quad (n, g, k) \in N \times G \times K.$$

Then we put

$$\text{Wh}(\pi, \eta, \tau)^{\text{mod}} = \bigcup_{i \in \text{Hom}_K(\tau^*, \pi|_K)} \{T_i \mid T \in \mathcal{I}_{\eta, \pi}, T_i \text{ is moderate growth}\}.$$

(Here the term "moderate growth" is by means of [9] §8.1.) According to the multiplicity one theorem of Shalika [8], the dimension of the space  $\text{Wh}(\pi, \eta, \tau)^{\text{mod}}$  is at most one. A unique (up to constant) element in  $\text{Wh}(\pi, \eta, \tau)^{\text{mod}}$  is called a *primary Whittaker function* with respect to  $(\pi, \eta, \tau)$ .

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3. WHITTAKER FUNCTIONS ON  $GL(3, \mathbb{C})$ 

In this section, we determine the primary Whittaker function on  $GL(3, \mathbb{C})$  for principal series representations and their minimal  $K$ -types.

**3.1. Groups and representations.** Let  $G = GL(3, \mathbb{C})$  be the complex general linear group of degree 3, which is viewed as a real reductive group, with the center

$$Z_G = \{r u 1_3 \mid r \in \mathbb{R}_{>0}, u \in U(1)\} \simeq \mathbb{C}^\times.$$

Here  $1_n$  is the unit matrix of degree  $n$ . Let  $K = U(3)$  be a maximal compact subgroup of  $G$ , and define subgroups  $A$  and  $N$  of  $G$  by

$$\begin{aligned} A &= \{\text{diag}(a_1, a_2, a_3) \in G \mid a_i \in \mathbb{R}_{>0}, i = 1, 2, 3\}, \\ N &= \left\{ n(x) = \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \in G \mid x = (x_i) \in \mathbb{C}^3 \right\}. \end{aligned}$$

Then we have an Iwasawa decomposition  $G = NAK$ . If we put

$$M = \{\text{diag}(u_1, u_2, u_3) \mid u_i \in U(1), i = 1, 2, 3\} \simeq U(1)^3,$$

then  $M$  is the centralizer of  $A$  in  $K$  and  $P = NAM$  gives the upper triangular subgroup of  $G$ , which is a minimal parabolic subgroup of  $G$ .

The equivalence classes of irreducible continuous representations of  $K$  are parameterized by the set of the highest weights

$$\Lambda = \{\mu = (\mu_1, \mu_2, \mu_3) \mid \mu \in \mathbb{Z}^3, \mu_1 \geq \mu_2 \geq \mu_3\}.$$

We denote by  $(\tau_\mu, V_\mu)$  the representation of  $K$  associated with  $\mu \in \Lambda$ . The representation space  $V_\mu$  has the (normalized) GZ-basis  $\{f(M)\}_{M \in G(\mu)}$  parameterized by the set  $G(\mu)$  of all  $G$ -patterns of type  $\mu$  (cf. [1], [2]). Here a  $G$ -pattern  $M \in G(\mu)$  is a triangle

$$M = \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ \alpha_1 & \alpha_2 & \\ \beta & & \end{pmatrix}$$

consisting of 6 integers satisfying the inequalities

$$\mu_1 \geq \alpha_1 \geq \mu_2 \geq \alpha_2 \geq \mu_3, \quad \alpha_1 \geq \beta \geq \alpha_2.$$

Let us take a character  $\sigma_n$  of  $M$  defined by

$$\sigma_n(\text{diag}(u_1, u_2, u_3)) = u_1^{n_1} u_2^{n_2} u_3^{n_3}, \quad n = (n_1, n_2, n_3) \in \mathbb{Z}^3,$$

and an element  $\nu$  in the dual  $\mathfrak{a}_\mathbb{C}^*$  of  $\mathfrak{a}_\mathbb{C}$  identified with  $(\nu_1, \nu_2, \nu_3) \in \mathbb{C}^3$  via  $\nu_i = \nu(E_{ii})$  for  $1 \leq i \leq 3$ . Here  $\mathfrak{a}_\mathbb{C}$  is the complexification of the Lie algebra of  $A$  and  $E_{ii}$  is the diagonal matrix unit with  $(i, i)$ -entry 1 and the remaining entries 0. Then the induced representation

$$\pi = \pi(\nu, \sigma_n) = \text{Ind}_P^G(1_N \otimes e^{\nu+\rho} \otimes \sigma_n)$$

of  $G$  from the parabolic subgroup  $P = NAM$  is called the *principal series representation* of  $G$ . Here  $\rho$  is the half-sum of the positive restricted roots, i.e.,

$$e^\rho(\text{diag}(a_1, a_2, a_3)) = \left(\frac{a_1}{a_3}\right)^2, \quad \text{diag}(a_1, a_2, a_3) \in A.$$

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The central character of  $\pi$  is given by

$$Z_G \ni r u l_3 \mapsto r^{\tilde{\nu}} u^{\tilde{n}}, \quad r \in \mathbb{R}_{>0}, u \in U(1),$$

with  $\tilde{\nu} = \nu_1 + \nu_2 + \nu_3$  and  $\tilde{n} = n_1 + n_2 + n_3$ , and the minimal  $K$ -type of  $\pi$  is the representation  $(\tau_{\mathbf{m}}, V_{\mathbf{m}})$  of  $K$  associated with the dominant permutation  $\mathbf{m} \in \Lambda$  of  $\mathbf{n}$ .

Finally, we take a non-degenerate character  $\eta$  of  $N$  defined by

$$\eta(n(\mathbf{x})) = \exp(2\pi\sqrt{-1}\operatorname{Im}(x_1 + x_3)).$$

**3.2. Differential equations.** Let us take an irreducible principal series representation  $\pi = \pi(\nu, \sigma_{\mathbf{n}})$  of  $G$  with the minimal  $K$ -type  $(\tau_{\mathbf{m}}, V_{\mathbf{m}})$  and a non-degenerate unitary character  $\eta$  of  $N$  defined in the previous subsection. In this subsection, we consider a system of differential equations for the functions  $\phi$  in  $\operatorname{Wh}(\pi, \eta, \tau)^{\operatorname{mod}}$ . This is described as that for the  $M$ -components  $\phi(M)$ , since the Whittaker functions are  $V_{\tau}$ -valued. Here the  $M$ -component  $\phi(M)$  of  $\phi$  corresponding to  $M \in G(\mathbf{m})$  is defined by

$$\phi(M; g) = \langle \phi(g), f(M) \rangle, \quad g \in G,$$

for the GZ-basis  $\{f(M)\}_{M \in G(\mathbf{m})}$  of  $V_{\mathbf{m}}$ .

It is well known that each element  $C$  in the center  $Z(\mathfrak{g}_{\mathbb{C}})$  of the universal enveloping algebra  $U(\mathfrak{g}_{\mathbb{C}})$  of  $\mathfrak{g}_{\mathbb{C}}$  acts as a scalar on the  $K$ -finite vectors in  $\pi$ . If we take an injection  $j \in \operatorname{Hom}_K(\tau_{\mathbf{m}}, \pi|_K)$ , then we have the equation

$$(1) \quad C \cdot j(f(M)) = \chi_C j(f(M)), \quad M \in G(\mathbf{m})$$

for a scalar  $\chi_C$ . Therefore, each  $M$ -component  $\phi(M)$  of  $\phi \in \operatorname{Wh}(\pi, \eta, \tau)^{\operatorname{mod}}$  satisfies the equations

$$(2) \quad C\phi(M) = \chi_C \phi(M), \quad C \in Z(\mathfrak{g}_{\mathbb{C}}).$$

Here we remark that the generators of  $Z(\mathfrak{g}_{\mathbb{C}})$  can be constructed from the Capelli elements in  $U(\mathfrak{g})$  (cf. [4]) via the identification of  $U(\mathfrak{g}_{\mathbb{C}})$  and  $U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{g})$ .

Let  $\mathfrak{k}$  (resp.  $\mathfrak{p}$ ) be the  $+1$  (resp. the  $-1$ ) eigenspace of the Cartan involution  $\theta$  of  $\mathfrak{g}$  defined by  $\theta(X) = -{}^tX$ . Then the complexification  $\mathfrak{p}_{\mathbb{C}}$  of  $\mathfrak{p}$  becomes a  $K$ -module via the adjoint action and its irreducible decomposition is  $\mathfrak{p}_{\mathbb{C}} = Z_{\mathfrak{p}, \mathbb{C}} \oplus \mathfrak{p}_{0, \mathbb{C}}$  with  $Z_{\mathfrak{p}, \mathbb{C}} \simeq V_{(0,0,0)}$  and  $\mathfrak{p}_{0, \mathbb{C}} \simeq V_{(1,0,-1)}$ . In the tensor product  $\mathfrak{p}_{0, \mathbb{C}} \otimes V_{\mu}$  with a general irreducible representation  $V_{\mu}$  of  $K$ ,  $V_{\mu}$  occurs with multiplicity two as the irreducible component. Take an injector  $\iota$  from  $V_{\mathbf{m}}$  into  $\mathfrak{p}_{0, \mathbb{C}} \otimes V_{\mathbf{m}} \simeq V_{(1,0,-1)} \otimes V_{\mathbf{m}}$  and fix an injection  $j \in \operatorname{Hom}_K(\tau_{\mathbf{m}}, \pi|_K)$ . Since the minimal  $K$ -type  $\tau_{\mathbf{m}}$  occurs with multiplicity one in  $\pi|_K$ , the composition

$$V_{\mathbf{m}} \xrightarrow{\iota} \mathfrak{p}_{0, \mathbb{C}} \otimes V_{\mathbf{m}} \xrightarrow{\alpha} \pi(\mathfrak{p}_{0, \mathbb{C}})j(V_{\mathbf{m}}) \subset L^2_{(M, \sigma_{\mathbf{n}})}(K)$$

is a scalar multiple of  $j$ , where  $\alpha$  is the evaluation map. Thus, if we write

$$\iota(f(M)) = \sum_{M' \in G(\mathbf{m})} X_{M, M'}^{(\iota)} \otimes f(M'), \quad X_{M, M'}^{(\iota)} \in \mathfrak{p}_{0, \mathbb{C}},$$

for the GZ-basis  $\{f(M)\}_{M \in G(\mathbf{m})}$  of  $V_{\mathbf{m}}$  then we have the equation

$$(3) \quad \sum_{M' \in G(\mathbf{m})} X_{M, M'}^{(\iota)} \cdot j(f(M')) = \lambda_{\iota} j(f(M)), \quad M \in G(\mathbf{m})$$

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with a scalar  $\lambda_\iota$ . We call this equation (3) *the Dirac-Schmid eigen-equation*. Then, for each injector  $\iota$ , we have the difference-differential equation

$$(4) \quad \sum_{M' \in G(\mathfrak{m})} X_{M, M'}^{(\iota)} \phi(M') = \lambda_\iota \phi(M),$$

among the  $M$ -components  $\{\phi(M)\}_{M \in G(\mathfrak{m})}$  of  $\phi \in \text{Wh}(\pi, \eta, \tau)^{\text{mod}}$ .

**3.3. Explicit integral formulas.** Let  $\pi = \pi(\nu, \sigma_n)$ ,  $(\tau^*, V_{\tau^*}) = (\tau_{\mathfrak{m}}, V_{\mathfrak{m}})$ , and  $\eta$  be as in the previous subsection. We give two explicit integral formulas for the primary Whittaker function  $\phi \in \text{Wh}(\pi, \eta, \tau)^{\text{mod}}$ , in this subsection.

The Whittaker functions are determined by its  $A$ -radial parts (i.e. its restriction to  $A$ ) because of the Iwasawa decomposition of  $G$ . Moreover, the values on the center  $Z_G$  of  $G$  are given by the central character of  $\pi$ , i.e.,

$$\phi(rug) = r^{\tilde{\nu}} u^{\tilde{n}} \phi(g), \quad \phi \in \text{Wh}(\pi, \eta, \tau)^{\text{mod}}, \quad r \in \mathbf{R}_{>0}, \quad u \in U(1), \quad g \in G.$$

Therefore, we can describe them as functions of two variables with the coordinates

$$y_1 = \frac{a_1}{a_2}, \quad y_2 = \frac{a_2}{a_3},$$

for  $\text{diag}(a_1, a_2, a_3) = a_3 \cdot \text{diag}(y_1 y_2, y_2, 1) \in A$ .

To state our results, we need some notations. If we write  $\mathfrak{m} = (n_a, n_b, n_c) \in \Lambda$ , then we put  $(\lambda_1, \lambda_2, \lambda_3) = \left( \nu_c - \frac{\tilde{\nu}}{3}, \nu_a - \frac{\tilde{\nu}}{3}, \nu_b - \frac{\tilde{\nu}}{3} \right)$ . For each  $G$ -pattern  $M = \begin{pmatrix} m_1 & m_2 & m_3 \\ \alpha_1 & \alpha_2 & \\ \beta & & \end{pmatrix} \in G(\mathfrak{m})$ , we put  $\delta(M) = \alpha_1 + \alpha_2 - m_2 - \beta$  and

$$\begin{aligned} \zeta_1^{(1)}(M) &= \lambda_1 - m_3 + \beta, & \zeta_1^{(2)}(M) &= -\lambda_1 + m_1 - \beta - \delta(M), \\ \zeta_2^{(1)}(M) &= \lambda_2 + m_1 - \beta, & \zeta_2^{(2)}(M) &= -\lambda_2 - m_3 + \beta + \delta(M), \\ \zeta_3^{(1)}(M) &= \lambda_3 + \alpha_1 - \alpha_2 - |\delta(M)|, & \zeta_3^{(2)}(M) &= -\lambda_3 + m_1 - m_3 - \alpha_1 + \alpha_2. \end{aligned}$$

Now we can state our main result, that is, two explicit integral formulas for the primary Whittaker function with respect to the triple  $(\pi, \eta, \tau)$ .

**Theorem 3.1.** *Let  $W_3(y)$  be the  $A$ -radial part of the primary Whittaker function in  $\text{Wh}(\pi, \eta, \tau)^{\text{mod}}$  and  $W_3(M; y) = y_1^2 y_2^2 \tilde{W}_3(M; y)$  be its  $M$ -component. Then  $\tilde{W}_3(M; y)$  has the following integral expressions:*

$$\begin{aligned} \tilde{W}_3(M; y) &= \frac{1}{(2\pi\sqrt{-1})^2} \int_{s_1} \int_{s_2} V_3(M; s_1, s_2) (\pi y_1)^{-s_1} (\pi y_2)^{-s_2} ds_1 ds_2 \\ &= 2^4 (\pi y_1)^{\frac{-\lambda_3 + m_1 - m_3}{2}} (\pi y_2)^{\frac{\lambda_3 + m_1 - m_3}{2}} \\ &\quad \times \int_0^\infty K_A \left( 2\pi y_1 \sqrt{1 + \frac{1}{v}} \right) K_{A+\delta(M)} \left( 2\pi y_2 \sqrt{1 + v} \right) v^B (1 + v)^C \frac{dv}{v}. \end{aligned}$$

Here, in the first integral expression of Mellin-Barnes type, the paths  $s_i$  of integrations are the vertical lines from  $\text{Re } s_i - \sqrt{-1}\infty$  to  $\text{Re } s_i + \sqrt{-1}\infty$  with enough large

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real part and the integrand  $V_3(M; s_1, s_2)$  is defined by

$$V_3(M; s_1, s_2) = \prod_{i=1}^2 \prod_{j=1}^3 \Gamma\left(\frac{s_i + \zeta_j^{(i)}(M)}{2}\right) / \Gamma\left(\frac{s_1 + s_2 + \zeta_3^{(1)}(M) + \zeta_3^{(2)}(M)}{2}\right).$$

Also, in the second integral expression of Euler type,  $K_\nu$  is the  $K$ -Bessel function and the parameters  $A$ ,  $B$  and  $C$  are given as follows.

$$A = \frac{\zeta_1^{(1)}(M) - \zeta_2^{(1)}(M)}{2}, \quad B = \frac{2\zeta_3^{(1)}(M) - \zeta_1^{(1)}(M) - \zeta_2^{(1)}(M)}{4}, \quad C = \frac{|\delta(M)|}{2}.$$

This theorem is obtained by solving the system of difference-differential equations (2) and (4) for the  $M$ -components of the Whittaker function in the previous subsection, explicitly.

**Remark 3.2.** The holonomic system of differential equations given in §3.2 has regular singularities along 2 divisors  $y_1 = 0$  and  $y_2 = 0$  which are of simple normal crossing at  $(y_1, y_2) = (0, 0)$ . The power series solutions of the system at  $(y_1, y_2) = (0, 0)$  are called the secondary Whittaker functions. The secondary Whittaker functions play an important role in constructing the Poincaré series (cf. [6], [7]). Our proof of the main theorem requires the factorization theorem of the primary Whittaker functions by the secondaries.

**Remark 3.3.** In the explicit description of the Dirac-Schmid eigen-equations, we used the Clebsch-Gordan coefficients for the injectors  $\iota : V_\mu \rightarrow V_\mu \otimes V_{(1,0,-1)}$  with respect to the GZ-basis. Our paper [2] discussed its dual, that is, the Clebsch-Gordan coefficients for the projectors from  $V_\mu \otimes V_{(1,0,-1)}$  to  $V_\mu$ .

## 4. PROPAGATION FORMULA FOR WHITTAKER FUNCTIONS

In this section, we give an expression of the primary Whittaker function on  $GL(3, \mathbb{C})$  in terms of that on  $GL(2, \mathbb{C})$ . This is an analogue of the formula obtained by Ishii-Stade [5].

**4.1. Whittaker functions on  $GL(2, \mathbb{C})$ .** First we recall two explicit integral formulas of the principal series Whittaker functions on  $GL(2, \mathbb{C})$ .

Let  $G' = GL(2, \mathbb{C})$  and take subgroups  $K' = U(2)$ , and

$$A' = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \mid a_i \in \mathbb{R}_{>0}, i = 1, 2 \right\}, \quad N' = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{C} \right\},$$

of  $G'$ . Then we have an Iwasawa decomposition  $G' = N'A'K'$  of  $G'$ . The upper triangular subgroup  $P' = N'A'M'$  of  $G'$  with the centralizer  $M'$  of  $A'$  in  $K'$  given by

$$M' = \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \mid u_i \in U(1), i = 1, 2 \right\} \simeq U(1)^2,$$

is the minimal parabolic subgroup.

We can parameterize the equivalence classes of irreducible continuous representations of  $K' = U(2)$  by the set

$$\Lambda' = \{\mu' = (\mu'_1, \mu'_2) \mid \mu' \in \mathbb{Z}^2, \mu'_1 \geq \mu'_2\},$$

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from the highest weight theory. The representation space  $V_{\mu'}$  of the representation  $\tau_{\mu'}$  associated with  $\mu' = (\mu'_1, \mu'_2) \in \Lambda'$  has the (normalized) GZ-basis  $\{f'(M')\}_{M' \in G(\mu')}$  as in the case of  $U(3)$ . Here

$$G(\mu') = \left\{ M' = \begin{pmatrix} \mu'_1 & \mu'_2 \\ \alpha' & \end{pmatrix} \mid \alpha' \in \mathbf{Z}, \mu'_1 \geq \alpha' \geq \mu'_2 \right\}.$$

A principal series representation

$$\pi' = \pi'(\nu', \sigma_{\mathbf{n}'}) = \text{Ind}_{\mathbf{P}'}^{G'}(1_{N'} \otimes e^{\nu' + \rho'} \otimes \sigma_{\mathbf{n}'}),$$

of  $G'$  with data  $\nu' = (\nu'_1, \nu'_2) \in \mathbf{C}^2$  and  $\mathbf{n}' = (n'_1, n'_2) \in \mathbf{Z}^2$  is defined similarly to that of  $GL(3, \mathbf{C})$ . Here the half-sum  $\rho'$  of the positive restricted roots is given by

$$e^{\rho'}(\text{diag}(a_1, a_2)) = \frac{a_1}{a_2}, \quad \text{diag}(a_1, a_2) \in A'.$$

As in the case of  $GL(3, \mathbf{C})$ , the central character of  $\pi'$  is

$$Z_{G'} = \{ru1_2 \mid r \in \mathbf{R}_{>0}, u \in U(1)\} \ni ru1_2 \mapsto r^{\tilde{\nu}'} u^{\tilde{n}'}, \quad r \in \mathbf{R}_{>0}, u \in U(1),$$

with  $\tilde{\nu}' = \nu'_1 + \nu'_2$  and  $\tilde{n}' = n'_1 + n'_2$ , and the minimal  $K'$ -type of  $\pi'$  is the representation  $(\tau_{\mathbf{m}'}, V_{\mathbf{m}'})$  associated with the dominant permutation  $\mathbf{m}' \in \Lambda'$  of  $\mathbf{n}'$ .

Also, we take a non-degenerate character  $\eta'$  of  $N'$  defined by

$$\eta'(n(x)) = \exp(2\pi\sqrt{-1}\text{Im}(x)).$$

By virtue of the Iwasawa decomposition of  $G'$  and the central character of  $\pi'$ , the Whittaker functions can be described as functions of a variable

$$y = \frac{a_1}{a_2}, \quad \text{for } \text{diag}(a_1, a_2) = a_2 \cdot \text{diag}(y, 1) \in A'.$$

**Theorem 4.1.** *Let  $\pi' = \pi'(\nu', \sigma_{\mathbf{n}'})$  be an irreducible principal series representation of  $G'$  with the minimal  $K'$ -type  $(\tau_{\mathbf{m}'}, V_{\mathbf{m}'})$  associated with the dominant permutation  $\mathbf{m}' = (m'_1, m'_2) = (n'_a, n'_b) \in \Lambda'$  of  $\mathbf{n}'$ , and let  $\eta'$  be a non-degenerate unitary character of  $N'$ . Moreover let  $W_2(y) \in \text{Wh}(\pi', \eta', \tau')^{\text{mod}}$  be the ( $A'$ -radial part of) primary Whittaker function with  $M'$ -components  $W_2(M'; y) = y\tilde{W}_2(M'; y)$  for each  $G$ -pattern  $M' = \begin{pmatrix} m'_1 & m'_2 \\ \alpha' & \end{pmatrix} \in G(\mathbf{m}')$  of weight  $(w'_1, w'_2) = (\alpha', m'_1 + m'_2 - \alpha')$ . Then the function  $\tilde{W}_2(M'; y)$  has the following expressions:*

$$\tilde{W}_2(M'; y) = \frac{1}{2\pi\sqrt{-1}} \int_s V_2(M'; s) (\pi y)^{-s} ds = 4(\pi y)^A K_B(2\pi y).$$

Here, the path of integration is the vertical line from  $\text{Re } s - \sqrt{-1}\infty$  to  $\text{Re } s + \sqrt{-1}\infty$  with enough large real part and the integrand  $V_2(M'; s)$  is defined by

$$V_2(M'; s) = \Gamma\left(\frac{s + \lambda'_2 + m'_1 - \alpha'}{2}\right) \Gamma\left(\frac{s + \lambda'_1 + \alpha' - m'_2}{2}\right),$$

with

$$\lambda'_1 = \nu'_b - \frac{\tilde{\nu}'}{2}, \quad \lambda'_2 = \nu'_a - \frac{\tilde{\nu}'}{2},$$

and the parameters  $A$  and  $B$  are given by

$$A = \frac{m'_1 - m'_2}{2}, \quad B = \frac{\lambda'_1 - \lambda'_2 + w'_1 - w'_2}{2}.$$

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This theorem is obtained by solving a system of differential equations for the Whittaker functions, as in the case of  $GL(3, \mathbb{C})$ . But the required calculation is much simpler than that of  $GL(3, \mathbb{C})$ .

**Remark 4.2.** *As in the case of  $GL(3, \mathbb{C})$ , we can show the factorization theorem of the primary Whittaker functions by the secondaries, which is essentially the expression of the  $K$ -Bessel function by the  $I$ -Bessel functions.*

**4.2. Some integral formulas.** The modified Bessel function  $K_\nu(z)$  of the second kind has several integral expressions. Among them, we recall two expressions: One is the integral expression of Mellin-Barnes type

$$K_\nu(z) = \frac{1}{4} \cdot \frac{1}{2\pi\sqrt{-1}} \int_s \Gamma\left(\frac{s+\nu}{2}\right) \Gamma\left(\frac{s-\nu}{2}\right) \left(\frac{z}{2}\right)^{-s} ds.$$

Here, the path of integration is the vertical line from  $\operatorname{Re} s - \sqrt{-1}\infty$  to  $\operatorname{Re} s + \sqrt{-1}\infty$  with enough large real part. Another is that of Euler type

$$K_\nu(z) = \frac{1}{2} \int_0^\infty \exp\left(\frac{-z(t+t^{-1})}{2}\right) t^\nu \frac{dt}{t},$$

which is valid only for  $\operatorname{Re} z > 0$ .

Also we recall the following integral formula so-called Barnes' lemma

$$\begin{aligned} & \frac{1}{2\pi\sqrt{-1}} \int_z \Gamma(z+a)\Gamma(z+b)\Gamma(-z+c)\Gamma(-z+d)dz \\ &= \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}. \end{aligned}$$

Here the path of integration is the vertical line from  $\operatorname{Re} z - \sqrt{-1}\infty$  to  $\operatorname{Re} z + \sqrt{-1}\infty$  with enough large real part.

**4.3. Propagation formula.** Let  $\pi = \pi(\nu, \sigma_{\mathbf{n}})$  be an irreducible principal series representation of  $G = GL(3, \mathbb{C})$  with data  $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{C}^3$  and  $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3$  and let  $\eta$  be a non-degenerate unitary character of  $N$  defined in §3. For simplicity, we assume that the parameter  $\mathbf{n}$  satisfies the regularity condition

$$n_1 \geq n_2 \geq n_3.$$

Then  $\mathbf{n} \in \Lambda$  and the minimal  $K$ -type of  $\pi$  is  $(\tau_{\mathbf{m}}, V_{\mathbf{m}}) = (\tau_{\mathbf{n}}, V_{\mathbf{n}})$ .

Let  $W_3(y) \in \operatorname{Wh}(\pi, \eta, \tau)^{\operatorname{mod}}$  be the ( $A$ -radial part of) primary Whittaker function with  $M$ -components  $W_3(M; y) = y_1^2 y_2^2 \tilde{W}_3(M; y)$ . Under the regularity condition on  $\mathbf{n}$ , we have the parameters  $(\lambda_1, \lambda_2, \lambda_3) = \left(\nu_3 - \frac{\tilde{\nu}}{3}, \nu_1 - \frac{\tilde{\nu}}{3}, \nu_2 - \frac{\tilde{\nu}}{3}\right)$  which appear in the integrand  $V_3(M; s_1, s_2)$  of the integral expression of Mellin-Barnes type for  $\tilde{W}_3(M; y)$  in Theorem 3.1.



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**Theorem 4.3.** Let  $M = \begin{pmatrix} m_1 m_2 m_3 \\ \alpha_1 \alpha_2 \\ \beta \end{pmatrix} \in G(\mathbf{m})$ . Then the integrand  $V_3(M; s_1, s_2)$  has the following expression.

$$V_3(M; s_1, s_2) = \Gamma\left(\frac{s_1 + \zeta_j^{(1)}(M)}{2}\right) \Gamma\left(\frac{s_2 + \zeta_j^{(2)}(M)}{2}\right) \\ \times \frac{1}{2\pi\sqrt{-1}} \int_z \Gamma\left(\frac{z + s_1 + \mu_1}{2}\right) \Gamma\left(\frac{z + s_2 + \mu_2}{2}\right) V_2(M'; -z) dz,$$

where  $V_2(M'; s)$  is the integrand of the integral expression of  $\tilde{W}_2(M'; y)$  in Theorem 4.1 for a triple  $(\pi'(\nu', \sigma_{\mathbf{n}'}), \eta', \tau_{\mathbf{m}'})$  and a  $G$ -pattern  $M' \in G(\mathbf{m}')$  and the path of integration is the vertical line from  $\operatorname{Re} z - \sqrt{-1}\infty$  to  $\operatorname{Re} z + \sqrt{-1}\infty$  with large enough real part. The parameters and the representations are given in the following table.

	$j$	$\mu_1$	$\mu_2$	$\nu'$	$\mathbf{n}'$	$M'$
$\delta(M) \geq 0$	2	$-\frac{\lambda_2}{2} - \alpha_2 + \beta$	$\frac{\lambda_2}{2} + m_1 - \alpha_1$	$(\nu_2, \nu_3)$	$(m_2, m_3)$	$\begin{pmatrix} m_2 m_3 \\ \alpha_2 \end{pmatrix}$
$\delta(M) \leq 0$	1	$-\frac{\lambda_1}{2} + \alpha_1 - \beta$	$\frac{\lambda_1}{2} + \alpha_2 - m_3$	$(\nu_1, \nu_2)$	$(m_1, m_2)$	$\begin{pmatrix} m_1 m_2 \\ \alpha_1 \end{pmatrix}$
$\delta(M) = 0$	3	$-\frac{\lambda_3}{2}$	$\frac{\lambda_3}{2}$	$(\nu_1, \nu_3)$	$(m_1, m_3)$	$\begin{pmatrix} m_1 m_3 \\ \beta \end{pmatrix}$

*Proof.* Assume  $\delta(M) \geq 0$ . Then, since  $\zeta_1^{(1)}(M) + \zeta_1^{(2)}(M) = \zeta_3^{(1)}(M) + \zeta_3^{(2)}(M)$ , Barnes' lemma leads the equation

$$V_3(M; s_1, s_2) = \Gamma\left(\frac{s_1 + \zeta_2^{(1)}(M)}{2}\right) \Gamma\left(\frac{s_2 + \zeta_2^{(2)}(M)}{2}\right) \\ \times \frac{1}{2\pi\sqrt{-1}} \int_z \Gamma\left(\frac{z + s_1 + \mu_1}{2}\right) \Gamma\left(\frac{z + s_2 + \mu_2}{2}\right) \\ \times \Gamma\left(\frac{-z + \mu_3}{2}\right) \Gamma\left(\frac{-z + \mu_4}{2}\right) dz,$$

where the parameters  $\mu_1$  and  $\mu_2$  are given in the assertion of theorem and  $\mu_3$  and  $\mu_4$  are

$$\mu_3 = \frac{-\nu_2 + \nu_3}{2} + \alpha_2 - m_3, \quad \mu_4 = \frac{\nu_2 - \nu_3}{2} - \alpha_2 + m_2.$$

Here we use the relations  $\lambda_1 + \frac{\lambda_2}{2} = \frac{-\nu_2 + \nu_3}{2}$  and  $\lambda_3 + \frac{\lambda_2}{2} = \frac{\nu_2 - \nu_3}{2}$ .

The assertion for the other cases of  $\delta(M)$  can be obtained similarly.  $\square$

**Corollary 4.4.** We have the following expression of  $\tilde{W}_3(M; y)$ .

$$\tilde{W}_3(M; y) = 4\pi^{a_1+a_2} y_1^{a_1+A_1} y_2^{a_2-A_2} \int_0^\infty \int_0^\infty \exp\left(-\pi\left(y_1^2 u_1 + \frac{1}{u_1} + y_2^2 u_2 + \frac{1}{u_2}\right)\right) \\ \times u_1^{A_1} u_2^{-A_2} \tilde{W}_2\left(M'; \pi y_2 \sqrt{\frac{u_2}{u_1}}\right) \frac{du_1}{u_1} \frac{du_2}{u_2}.$$

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Here

$$a_k = \frac{1}{2} \left\{ \zeta_j^{(k)}(M) + \mu_k \right\}, \quad A_k = \zeta_j^{(k)}(M) - a_k, \quad k = 1, 2,$$

and the parameters and the representations are given in Theorem 4.3.

*Proof.* This corollary is obtained from the integral expression of Mellin-Barnes type for  $W_3(M; y)$  in Theorem 3.1 and Theorem 4.3 by using the integral expressions of  $K_\nu(z)$  in §4.2.  $\square$

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